# RESEARCH NOTES

## A Note on Reinforced Incomplete Block Designs

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Das (1954) discussed an incomplete block design in which "the incompleteness is partial but completely balanced". It was obtained by adding a set of $\alpha$ new treatments in every block of a BIB design ($v$, $k$, $r$, $b$, $\lambda$) and also by adding $\beta$ complete blocks, *i.e.*, each containing $v + \alpha$ treatments so that the new design had blocks of two sizes, namely, $k + \alpha$ and $v + \alpha$ plots.

When $\beta = 0$ in Das' design it becomes a special case of the Intra- and Inter-Group balanced incomplete block design introduced by Nair and Rao (1942). The parameters of the latter assume the following values in the case of Das' design.

$$\left.\begin{array}{lll} \lambda_{11} = \lambda, & \lambda_{12} = r, & \lambda_{22} = b \\ v_1 = v, & v_2 = \alpha, & (k) = k + \alpha \\ r_1 = r, & r_2 = b & \end{array}\right\} \qquad (1)$$

One point of difference between Nair and Rao's and Das' methods for obtaining the estimate of the effect of each of the original set of $v$ treatments and of each of the new set of $\alpha$ treatments should be noted at this stage. Nair and Rao used the constraining relation:—

$$r_1 \sum_1^{v_1} v_{1i} + r_2 \sum_1^{v_2} v_{2i} = 0 \qquad (2)$$

and obtained the following intra-block estimates for $v_{1i}$ and $v_{2i}$:

$$v_{1i} = \frac{k}{v_1\lambda_{11} + v_2\lambda_{12}} \left\{ Q_{1i} + \frac{1}{bk\lambda_{12}} (r_2\lambda_{11} - r_1\lambda_{12}) \sum_1^{v_1} Q_{1i} \right\} \qquad (3)$$

$$v_{2i} = \frac{k}{v_1\lambda_{12} + v_2\lambda_{22}} \left\{ Q_{2i} + \frac{1}{bk\lambda_{12}} (r_1\lambda_{22} - r_2\lambda_{12}) \sum_1^{v_2} Q_{2i} \right\} \qquad (4)$$

where $Q$ for a particular treatment is its total yield *minus* sum of the means of blocks in which it appears.

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The constraining relation used by Das for his design corresponds to

$$\sum_{1}^{v_1} v_{1i} + \sum_{1}^{v_2} v_{2i} = 0 \tag{5}$$

If relation (5) had been used for the Intra- and Inter-Group balanced designs, the intra-block estimates of $v_{1i}$ and $v_{2i}$ would have been

$$v_{1i} = \frac{k}{v_1\lambda_{11} + v_2\lambda_{12}} \left\{ Q_{1i} + \frac{1}{v\lambda_{12}} (\lambda_{11} - \lambda_{12}) \sum_{1}^{v_1} Q_{1i} \right\} \tag{6}$$

$$v_{2i} = \frac{k}{v_1\lambda_{12} + v_2\lambda_{22}} \left\{ Q_{2i} + \frac{1}{v\lambda_{12}} (\lambda_{22} - \lambda_{12}) \sum_{1}^{v_2} Q_{2i} \right\} \tag{7}$$

By substituting in (6) and (7) the set of relations given in (1) we obtain Das' estimates for reinforced BIB designs (when $\beta = 0$), namely,

$$v_{1i} = \frac{k + a}{v\lambda + ra} \left\{ Q_{1i} + \frac{(\lambda - r)}{r(v + a)} \sum_{1}^{v} Q_{1i} \right\} \tag{8}$$

$$(i = 1, 2, \ldots, v)$$

$$v_{2i} = \frac{1}{b} \left\{ Q_{2i} + \frac{(b - r)}{r(v + a)} \sum_{1}^{a} Q_{2i} \right\} \tag{9}$$

$$(i = 1, 2, \ldots, a)$$

Whatever be the linear constraining relation used, be it (2), (5) or any other, it is easy to see that estimates of $v_{1i} - v_{1i'}$, $v_{2i} - v_{2i'}$ and $v_{1i} - v_{2i'}$ and hence of their variances as also the sum of squares due to treatments adjusted for blocks would be invariant to the constraining relation. Constraining relation (2) is, however, to be preferred because of its advantage in recovering the inter-block information using the simple rule devised by Rao (1947) [*see* Equations (3.9) to (3.13) of his paper and the discussion just following them].

In order that we may recover the inter-block information for reinforced BIB designs using Rao's rule, we will have to obtain fresh solutions for intra-block estimates of the treatment effects based on the constraining relation (2), *i.e.*, by substituting the relations (1) in the expressions (3) and (4). We then obtain,

$$v_{1i} = \frac{1}{v\lambda + ra} \left\{ (k + a)\, Q_{1i} - \frac{(b - r)}{b\,(v - 1)} \sum_{1}^{v} Q_{1i} \right\} \tag{10}$$

$$(i = 1, 2, \ldots, v)$$

$$v_{2i} = \frac{1}{b}\, Q_{2i} \qquad (i = 1, 2, \ldots, a) \tag{11}$$

It will be admitted that (11) is much simpler than (9) but constraining relation (5) was a necessity with Das to cover designs with $\beta \neq 0$.

Using Rao's rule, it could be seen that inter-block information does not exist for $v_{2i}$ and that the estimate of $v_{1i}$ after recovery of inter-block information is

$$\hat{v}_{1i} = \frac{\left\{ (k+a)\,(wQ_{1i} + w'Q'_{1i}) - \frac{(b-r)\,(w-w')}{b\,(v-1)\,w} \left( w \sum_{1}^{v} Q_{1i} + w' \sum_{1}^{v} Q'_{1i} \right) \right\}}{\{(v\lambda + ra)\, w + (r - \lambda)\, w'\}} \tag{12}$$

It could also easily be verified from (10) and (12) that estimate of $v_{1i} - v_{1i'}$ can be obtained by replacing $k$ by $k + a$ in the corresponding estimates for $v_{1i} - v_{1i'}$ in the original BIB design $(v, k, r, b, \lambda)$. It therefore follows that the expression for the variance between any pair of the original set of $v$ treatments in the reinforced BIB design, *i.e.*, $V(v_{1i} - v_{1i'})$ can be obtained by replacing $k$ by $k + a$ in the expression for this variance in the original BIB design. Thus, for intra-block analysis, the variance of $v_{1i} - v_{1i'}$ in the original BIB design is

$$\frac{2k\sigma_k^2}{r\,(k - 1) + \lambda} = \frac{2k\sigma_k^2}{v\lambda} \tag{13}$$

where $\sigma_k^2$ is the variance per plot within blocks of $k$ plots.

Replacing $k$ by $k + a$ in (13) we get for the reinforced BIB design the intra-block variance of $v_{1i} - v_{1i'}$ to be

$$V(v_{1i} - v_{1i'}) = \frac{2\,(k + a)\,\sigma^2_{k+a}}{r\,(k + a - 1) + \lambda}$$

$$= \frac{2\,(k + a)\,\sigma^2_{k+a}}{v\lambda + ra} \tag{14}$$

where $\sigma^2_{k+a}$ is the variance per plot within blocks of $k + a$ plots.

The relative efficiency of intra-block contrasts among the $v$ original treatments in the reinforced BIB design compared to the parent BIB design is given by

$$\frac{\sigma_k^2}{\sigma^2_{k+\alpha}} \cdot \frac{1+\dfrac{r\alpha}{v\lambda}}{1+\dfrac{\alpha}{k}}. \tag{15}$$

Though this relative efficiency need not necessarily exceed unity (and often may not) the efficiency factor

$$\left(1+\frac{r\alpha}{v\lambda}\right) \Big/ \left(1+\frac{\alpha}{k}\right) \tag{16}$$

is greater than unity as has also been shown by Das (1954). For this reason the $\alpha$ new treatments may be said to reinforce the parent BIB design inasmuch as they tend to make the comparisons among the set of $v$ original treatments more accurate.

Giri (1957) applied the same reinforcement of $\alpha$ new treatments and $\beta$ complete blocks to PBIB designs and indicated how the intra-block analysis could be done for the two-associate case. He used the constraining relation

$$\sum_1^v v_i + \sum_1^\alpha v_{i'} = 0 \tag{17}$$

where $v_i$ and $v_{i'}$ represent the effect of the $i$-th and $i'$-th treatment respectively among the original set of $v$ treatments and the new set of $\alpha$ treatments.

In the case of Giri's design also, when $\beta = 0$ and the number of associate classes in the parent PBIB design is $m$ ($\geqslant 2$), both the intra- and inter-block analysis could be done by the method described by Rao (1947) in Section 3 of his paper if the constraining relation used is

$$r \sum_1^v v_i + b \sum_1^\alpha v_{i'} = 0 \tag{18}$$

and not (17). It will then be found that the intra-block estimate of $v_{i'}$ is, like (11), a very simple expression, namely,

$$v_{i'} = \frac{1}{b} Q_{i'} \qquad (i' = 1, 2, \ldots, \alpha) \tag{19}$$

and that no inter-block information exists for $v_{i'}$, *i.e.*, for any of the $\alpha$ new treatments. Furthermore, expressions for the intra-block and the combined intra- and inter-block estimates of the differences among the set of $v$ original treatments and their variances could be obtained from the corresponding expressions for the parent PBIB design ($v, k, r, b, \lambda_i, n_i, p_{jk}^i$) by replacing $k$ by $k + \alpha$ in those expressions. This will automatically mean replacing $A_{12}$ and $B_{22}$ defined in (3.51) of Bose and Nair's (1939) paper by $A_{12} + r\alpha$ and $B_{22} + r\alpha$ respectively for two-associate PBIB designs and replacing $A_{13}$, $B_{23}$ and $C_{33}$ defined in (3.102) of that paper by $A_{13} + r\alpha$, $B_{23} + r\alpha$ and $C_{33} + r\alpha$ respectively for three-associate PBIB designs. Similar changes are required in $A_{14}$, $B_{24}$, $C_{34}$ and $D_{44}$ defined in (29) of Nair's (1952) paper for four-associate PBIB designs.

Giri (1957) has given a proof, due to M. N. Das, that the variances of differences between pairs among the original set of $v$ treatments in the reinforced two-associate PBIB design (with $\alpha > 0$, $\beta = 0$) will be less than the corresponding variance in the parent two-associate PBIB design ($\alpha = 0$, $\beta = 0$) assuming $\sigma_{k+\alpha} = \sigma_k$. An alternative proof is given below:—

Consider the variance of difference between two second associates among the original set of $v$ treatments in the two designs. The coefficients of $\sigma_k^2$ and $\sigma^2_{k+\alpha}$ respectively in the expressions for these variances are:—

For PBIB:

$$\frac{2kB_{22}}{A_{12}B_{22} - A_{22}B_{12}}. \tag{20}$$

For reinforced PBIB ($\alpha > 0$, $\beta = 0$):

$$\frac{2(k+\alpha)(B_{22}+r\alpha)}{(A_{12}+r\alpha)(B_{22}+r\alpha) - A_{22}B_{12}}. \tag{21}$$

We have to prove that (20) > (21).

It can be seen after a little simplification that the numerator of (20) *minus* (21) consists of the expression:

$$2\alpha \{B_{22}^2 (r - \lambda_2) + A_{22}B_{12}(rk + B_{22})\} + 2r\alpha^2 \{B_{22}(r - \lambda_2) + A_{22}B_{12}\} \tag{22}$$

where every term is positive. Since the denominator of (20) *minus* (21) is also positive, it follows that (20) > (21).

The corresponding coefficients of $\sigma_k^2$ and $\sigma^2_{k+a}$ in the expressions for the variances of difference between two first associates among the original set of $v$ treatments in the two designs are:—

For PBIB:

$$\frac{2k\,(B_{22} + B_{12})}{A_{12}B_{22} - A_{22}B_{12}}. \tag{23}$$

For reinforced PBIB ($a > 0,\ \beta = 0$):

$$\frac{2\,(k + a)\,(B_{22} + B_{12} + ra)}{(A_{12} + ra)\,(B_{22} + ra) - A_{22}\,B_{12}}. \tag{24}$$

We have to prove that (23) > (24).

Numerator of (23) *minus* (24) will contain, in addition to the expression (22), the following:—

$$2B_{12}a\,\{B_{22}\,(r - \lambda_2) + rkA_{12} + A_{22}B_{12}\} + 2kB_{12}r^2a^2. \tag{25}$$

All terms in (25) are positive. Hence (23) > (24).

Das (1958) has considered the case of reinforcing with $a$ new treatments and $\beta$ complete blocks any incomplete block design $(v, k, r, b, \lambda_{ij})$ where any pair $(i, j)$ out of the $v$ treatments occurs together in $\lambda_{ij}$ blocks. When $\beta = 0$, Das' reinforced incomplete block design is a special case of the general incomplete block design $(v, k, b, r_i, \lambda_{ij})$, whose analysis has been given by Rao (1947). If in the latter we replace $v$ and $k$ by $v + a$ and $k + a$ respectively and if

$$\left.\begin{aligned} r_i &= r\ (i = 1, 2, \ldots, v) \\ &= b\ (i = v + 1, v + 2, \ldots, v + a) \\ \lambda_{ij} &= \lambda_{ij}\ (i, j = 1, 2, \ldots, v) \\ &= r\ (i = 1, 2, \ldots, v;\ j = v + 1, v + 2, \ldots, v + a) \\ &= b\ (i, j = v + 1, v + 2, \ldots, v + a) \end{aligned}\right\} \tag{26}$$

we get Das' design for the case $\beta = 0$. The intra- and inter-block estimates of differences among the $v$ original treatments, their variances, etc., can also be obtained directly from the results in Section 3 of Rao's paper by putting $r_i = r$ in those results and then replacing $k$ by $k + a$ and using the constraining relation (18).

Analysis of a reinforced incomplete block design more general than the one considered by Das (1958) can be obtained from the analysis given by Rao for the design $(v, k, b, r_i, \lambda_{ij})$ by reinforcing it with $a$

new treatments and without adding $\beta$ complete blocks. The analysis of this general reinforced incomplete block design follows from Rao's results by replacing $k$ by $k + a$ and using the constraining relation:

$$\sum_{1}^{v} r_i v_i + b \sum_{1}^{a} v_{i'} = 0. \tag{27}$$

Finally, I should add that the reinforced incomplete block design when $\beta = 0$ is an interesting extension of the old practice of "Method of Controls" in varietal trials discussed by Yates (1936) in Section II of his paper.

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# A Research Note on "The Variance of the Coefficient of Genetic Correlation Estimated in Plant Breeding Experiments"


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THE coefficient of genetic correlation between two characters measures the degree of association between the genetic values for them and its estimation is fairly well known (Fairfield Smith, 1936; Lush, 1949). Reeve (1955) gives a method of estimation of variance of genetic correlation coefficient from data on animal parents and offspring. The

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same procedure cannot however be applied in plant breeding since the method of estimation of genetic correlation is different. In this note, an approximate formula, for large number of observations for estimating the variance of genetic correlation coefficient estimated from plant breeding experiments is presented.

Suppose $v$ pure bred varieties or strains are tried in a randomised block design with $b$ blocks and $x$ and $y$ are the two characters observed in the experiment for which the genetic correlation is to be estimated. Let the analysis of variance and covariance of the two characters be symbolically put as shown below in Table I.

TABLE I

*Analysis of Variance and Covariance of* x *and* y

| Source of variation | D.F. | Mean square (M.S.) for $x$ | Expected value of M.S. for $x$ | Mean product (M.P.) between $x$ and $y$ | Expected value of M.P. | Mean square (M.S.) for $y$ | Expected value of M.S. for $y$ |
|---|---|---|---|---|---|---|---|
| Varieties .. | $v-1$ | $v_{xx}$ | $V_{xx}$ | $v_{xy}$ | $V_{xy}$ | $v_{yy}$ | $V_{yy}$ |
| Error .. | $(b-1)(v-1)$ | $e_{xx}$ | $E_{xx}$ | $e_{xy}$ | $E_{xy}$ | $e_{yy}$ | $E_{yy}$ |

Assuming that the variation due to genetic differences in the error component is negligible, the true genetic variance of a character is estimated by the difference between the mean squares for varieties and error divided by the number of replications in the design. The genetic covariance between the two characters can be estimated similarly. Thus, symbolically, the genetic correlation $r_{gxy}$ between $x$, $y$, is given by $C/\sqrt{AB}$ where

$$A = v_{xx} - e_{xx}$$

$$B = v_{yy} - e_{yy}$$

$$C = v_{xy} - e_{xy}.$$

Applying the variance formula for large number of observations,

$$\frac{1}{r_{gxy}^2} \operatorname{Var}(r_{gxy})$$

$$= \frac{1}{C^2} \operatorname{Var}(C) + \frac{1}{4A^2} \operatorname{Var}(A) + \frac{1}{4B^2} \operatorname{Var}(B)$$

$$- \frac{1}{AC} \operatorname{Cov}(AC) - \frac{1}{BC} \operatorname{Cov}(BC) + \frac{1}{2AB} \operatorname{Cov}(AB) \tag{1}$$

The variances of $A$, $B$ are well known and the variance of the covariance term and covariance of covariance terms given in (1) are evaluated making use of the following formula (Nanda, 1949).

If $s_{ij}$ is an estimate of $\sigma_{ij}$ with $f$ degrees of freedom,

$$E(s_{ij} - \sigma_{ij})(s_{kl} - \sigma_{kl}) = (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})/f.$$

Substituting these variance and covariance expressions, we thus have

$$\begin{aligned} &\frac{1}{r_{gxy}^2} \operatorname{Var}(r_{gxy}) \\ &= \frac{1}{C^2}\left[\frac{V_{xy}^2 + V_{xx}V_{yy}}{v-1} + \frac{E_{xy}^2 + E_{xx}E_{yy}}{(b-1)(v-1)}\right] \\ &\quad + \frac{1}{2A^2}\left[\frac{V_{xx}^2}{v-1} + \frac{E_{xx}^2}{(b-1)(v-1)}\right] \\ &\quad + \frac{1}{2B^2}\left[\frac{V_{yy}^2}{v-1} + \frac{E_{yy}^2}{(b-1)(v-1)}\right] \\ &\quad - \frac{2}{AC}\left[\frac{V_{xx}V_{xy}}{v-1} + \frac{E_{xx}E_{xy}}{(b-1)(v-1)}\right] \\ &\quad - \frac{2}{BC}\left[\frac{V_{yy}V_{xy}}{v-1} + \frac{E_{yy}E_{xy}}{(b-1)(v-1)}\right] \\ &\quad + \frac{1}{AB}\left[\frac{V_{xy}^2}{v-1} + \frac{E_{xy}^2}{(b-1)(v-1)}\right] \qquad (2) \end{aligned}$$

Replacing the population values on right side of (2) by their estimates and making use of their relationships with the heritabilities and phenotypic correlation of the two characters, we have, after simplifying the algebra,

$$\begin{aligned} &\text{Estd Var}(r_{gxy}) \\ &= \frac{1}{(b-1)(v-1)}\left[2b\left(\frac{r_g}{H} - \frac{r_p}{G}\right)^2 + \frac{b}{G^2}\right. \\ &\quad \times \{(1 - r_g^2)(1 - r_p^2) + 4r_p^2\} + (1 - r_g^2) \\ &\quad \left. \times \left\{(1 - r_g^2) - \frac{2}{H} + \frac{2r_g r_p}{G}\right\}\right]. \qquad (3) \end{aligned}$$

Where

$G$ is the geometric mean of the estimates of the heritabilities of the two characters, $h_x^2$, $h_y^2$.

$H$ is the harmonic mean of the estimates of the heritabilities of the two characters.

$r_g$ is the estimated genetic correlation between the two characters.

$r_p$ is the estimated phenotypic correlation between the two characters.

The above method of estimating genetic correlation with its standard error is illustrated with the data on halo length ($y$) and fineness ($x$) of 19 strains of cotton tried in a randomised block design with four replications. The analysis of variance and covariance of the two characters is as shown below in Table II.

TABLE II

*Analysis of Variance and Covariance of Halo Length and Fineness of 19 Strains of Cotton*

| Source of Variation | D.F. | M.S. $y$ | M.P. $xy$ | M.S. $x$ |
|---|---|---|---|---|
| Blocks .. .. | 3 | 0·35 | —0·04 | 0·01 |
| Strains .. .. | 18 | 18·28 | —1·11 | 0·18 |
| Error .. .. | 54 | 1·22 | —0·01 | 0·08 |

From the above table, we have

$$r_g = -0\cdot 84, r_p = -0\cdot 61$$

$$h_x^2 = 0\cdot 56, h_y^2 = 0\cdot 93$$

$$\text{Var}(r_g) = 0\cdot 2539.$$

Thus the genetic correlation between halo length and fineness is—0·84 with an estimated standard error of 0·50.

I am grateful to Shri V. N. Amble for suggesting the present investigation and going through the manuscript.

I am thankful to Shri S. G. Pathak who kindly provided the data collected at the Indian Agricultural Research Institute, New Delhi, in the course of his work for his associateship.

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